AIAA Journal

VOLUME 3 AUGUST 1965 NUMBER 8

Review of Theoretical Investigations on Effect of Heat Transfer on Laminar Separation

MORRIS MORDUCHOW
Polytechnic Institute of Brooklyn, Brooklyn, N. Y.

Nomenclature

b_1 = coefficient of η in stagnation enthalpy profiles (Refs. 7 and 8)
H = stagnation enthalpy
h = H_w/H_1(for $Pr = 1$, $h = T_w/T_e$)
L = characteristic length
M = Mach number
Pr = Prandtl number
S = H/H_1
T = absolute temperature
T_e = equilibrium wall temperature for zero heat transfer
t = variable defined by Eq. (1)
U = transformed longitudinal velocity component according to Illingworth-Stewartson transformation
u, v = velocity components in x and y directions, respectively
X = transformed x variable according to Illingworth-Stewartson transformation
x, y = coordinates along and normal to wall, respectively
γ = ratio of specific heats
δ_t = boundary-layer thickness in (x, t) plane
η = t/δ_t
λ = $(\delta_t/L)^2(\rho_\infty u_\infty L/\mu_\infty)$
μ = coefficient of viscosity
ξ = x/L
ρ = mass density
τ = shear stress
φ = $-(\rho_w v_w/\rho_\infty u_\infty)(\rho_\infty u_\infty L/\mu_\infty)^{1/2}$

Subscripts

1 = value at local outer edge of boundary layer
∞ = value at reference point outside of boundary layer
s = value at separation point
w = value at wall
$(\ )'$ = denotes derivative with respect to ξ

I. Introduction

A REVIEW is given here of investigations on the effect of heat transfer on laminar separation. Although the literature on heat transfer is quite appreciable, the present review is concerned only with those papers that include *specifically* either research, numerical examples, or remarks on separation in steady, two-dimensional laminar boundary layers with heat transfer. Moreover, only "classical" boundary-layer theory, in which the local flow at the outer edge of the boundary layer is considered as prescribed, is considered. Thus, problems concerned with interaction between the boundary layer and the external flow are excluded here. Most of the investigations on separation with heat transfer pertain to an impermeable wall, but some results for a porous wall (with air-to-air injection or suction) have also been obtained.

It will be seen that, although the literature pertaining specifically to separation is perhaps rather small in comparison with the total literature on laminar boundary layers with heat transfer, it is nevertheless substantial. It will also be seen that appreciable room is left for further research in this area. It is noteworthy, for example, that a precise determination of the separation point in supersonic flows appears to be surprisingly difficult. In fact, for the special case of zero heat transfer and a linearly diminishing external velocity, although it is generally agreed that an increase in Mach number, with the fixed external velocity, moves the separation point upstream, there would still appear to be no definite universally accepted "exact" values for the location of the separation point for Mach numbers above zero. This is true, despite the fairly large number of calculations that have been

Morris Morduchow is Professor of Applied Mechanics in the Department of Aerospace Engineering and Applied Mechanics at the Polytechnic Institute of Brooklyn (PIB). He received the B.A. degree from Brooklyn College in 1942, majoring in mathematics, and then the B.Ae., M.Ae.E., and D.Ae.E. (1947) degrees from Polytechnic Institute of Brooklyn. He has been engaged in teaching and research at PIB since 1944, where he has held positions in both the aerospace engineering and mathematics departments. He is an Associate Fellow Member of the AIAA and a member of various scientific and honor societies. His research, which has been performed both independently and under sponsorship by the NACA, U. S. Air Force, and Office of Naval Research, has been principally in the fields of fluid dynamics, vibration theory, and numerical analysis.

Received March 29, 1965; revision received June 4, 1965. The study was supported by the Air Force Office of Scientific Research Grant No. AF-AFOSR-1-63. The author expresses his thanks to Martin H. Bloom for his interest in this work and his valuable discussions.

performed for this case. A summary of the various results that have been obtained for this zero-heat-transfer case is given in the Appendix. In this review the term "separation point" will be defined throughout as the point at the wall where the skin friction vanishes.

A brief chronological survey to give an over-all view of the pertinent literature will be presented first, and then a detailed systematic review of the various results will be given.

II. Brief Chronological Survey

The earliest investigation on laminar separation with heat transfer according to classical boundary-layer theory appears to have been made by Gadd[1] in 1952. He applied a finite-difference method, using Crocco variables x and u as independent variables and shear stress τ and temperature T as dependent variables. Calculations, however, were carried out with rather large steps in x. Morduchow and Galowin[2] made a fairly general analysis of the compressible laminar boundary layer with normal fluid injection, including the effect of wall temperature and injection on separation. However, the analysis was based on an application of the Kármán-Pohlhausen method with fourth degree profiles. Illingworth[3] used a series method in conjunction with a type of approximation made by Lighthill. The analysis was fairly involved, and some problems of convergence were encountered. Cohen and Reshotko[4] and Li and Nagamatsu[5] obtained the now well-known similarity solutions in compressible flows with heat transfer, and included certain results and conclusions pertinent to separation. Livingood and Donoughe[6] have made a convenient summary of low-speed similarity solutions with and without fluid injection and with constant or variable wall temperatures. Again, certain results contained therein are pertinent to separation. Morduchow and Grape[7] made an investigation of separation of the compressible laminar boundary layer with heat transfer based on an integral method in conjunction with sixth- and seventh-degree velocity profiles, and seventh-degree stagnation enthalpy profiles (cf. also Ref. 8). The computations were considerably simplified by using an approximate solution (essentially the zero-pressure-gradient solution) of the thermal equation to obtain b_1, the coefficient in the stagnation enthalpy profile proportional to the heat transfer at the wall. Morduchow[9] then extended the analysis of Ref. 7 to investigate laminar separation over a sweat-cooled wall taking into account a heat balance between the injected cooling air and the air in the boundary layer at the wall. In 1956 Gadd[10] made a concise review of investigations, to that date, on the effect of heat transfer on laminar separation. Baxter and Flügge-Lotz[11] have included in their applications of finite-difference methods certain problems pertinent to separation with heat transfer. Curle[12] applied Howarth's transformation for compressible flow, in conjunction with the assumption that the stagnation enthalpy is a quadratic function of the velocity u. Using also certain relations among the parameters that Thwaites had assumed for incompressible flow, Curle calculated a number of cases of separation with heat transfer. In 1960 Poots[13] considered the equations of flow in the transformed incompressible plane (by Stewartson's transformation) and carried out series solutions (in the manner of Howarth) for certain special cases corresponding to a heated wall and a wall with no heat transfer. These solutions may apparently be regarded as "exact." Poots also developed an extension of Tani's two-parameter integral method[14] to heat transfer. Luxton and Young[15] have presented what appears to be a fairly accurate, though somewhat involved, method (called their "complete" method) for calculating the compressible boundary layer with heat transfer and pressure gradient. They also show simplifications of this method; however, for certain problems involving separation, it appeared that only the "complete" method was sufficiently accurate. Curle[16] has modified his method of 1958 to account approximately for a Prandtl number other than one and a viscosity law other than a strict proportionality to temperature. (Reference 16 is concerned especially with interaction of a boundary layer and a shock wave in supersonic flow.) Monaghan[17] has developed a method for the compressible laminar boundary layer with heat transfer based on a combination of the Illingworth-Stewartson transformation,[18,19] Cohen and Reshotko's generalized method,[20] and Thwaites' method.[21] Certain specific calculations involving separation are included there. To investigate the effect of heat transfer on laminar separation, Savage,[22] like Poots, working in the transformed incompressible plane, has recently adapted Tani's method to flows with heat transfer, but assumes for all pressure gradients a universal normalized stagnation enthalpy profile, namely, that based on flow over a flat plate. This latter assumption (which appears to correspond somewhat to the simplifying approximation, previously referred to, of Morduchow and Grape[7] in solving the thermal equation) is justified in Ref. 22 by noting that, if the stagnation enthalpy profiles for separation according to the similarity solutions of Ref. 4 for various wall temperatures are plotted in a suitably "normalized" form, they all lie close to a single curve. In the integral method of Ref. 22, the velocity profiles are assumed as a linear combination of the Blasius profile and a separation type of profile based on the similarity solutions. Morduchow and Reyle[23] have included in their recent report calculations indicating the effect of wall cooling on boundary-layer separation with suction. However, the same type of approximation as previously indicated[7] for the solution of the thermal equation was made. In a recent publication,[24] using an integral type of method in conjunction with temperature profiles as quadratic functions of the velocity, Walz includes numerical examples indicating the effect of wall cooling on separation. Finally, Libby and Fox[44] have recently given and applied a multiple-parameter integral method, using Levy-Lees variables, in which additional equations are obtained by integrating the momentum and thermal equations multiplied by powers of the normal distance variable.

III. Some General Qualitative Conclusions

Before discussing specific numerical results that have been obtained, a number of general qualitative conclusions on laminar separation with heat transfer which have been established will first be reviewed.

Effects of Wall Temperature

The following conclusions on the effects of wall temperature on laminar skin friction and separation have been reached (more or less independently) by various investigators through different types of analyses: 1) cooling of the wall tends to lessen the direct effect* of a pressure gradient; 2) cooling of the wall tends to delay separation; and 3) cooling of the wall tends to diminish the skin friction in a favorable pressure gradient, but to increase the skin friction in an adverse pressure gradient. These conclusions were all reached in Ref. 2 on the basis of a Kármán-Pohlhausen type of analysis with fourth-degree velocity profiles and were subsequently confirmed by the use of higher degree profiles.[7] Moreover, they have also been reached on the basis of similarity solutions,[4,25] through the use of the Illingworth-Stewartson transformation,[26] and by the analyses of Illingworth,[3] Luxton and Young,[15] and Low.[27] Low, by his small-perturbation solutions, also notes that with sufficient cooling an adverse pressure gradient will tend to increase, instead of (as ordinarily) decrease, the skin friction.

Although the reader is referred to the references just cited for details on the establishment of conclusions 1–3, it may be

* By "direct" effect is meant here the influence of the gradient term $\partial p/\partial x$ proportional to du_1/dx. The influence of a pressure gradient, however, appears "indirectly" also, namely, through the dependence of u_1 and T_1 on x (cf., e.g., Refs. 2 and 7).

worthwhile to indicate briefly how at least conclusions 1 and 2 might be reached. Perhaps one of the quickest means is to note that by the Illingworth-Stewartson transformation [4,18–20,26] the momentum boundary-layer equation for compressible flows (with $\mu \sim T$, and $Pr = 1$) is reduced to the same form as that for incompressible flow, with the new pressure gradient term multiplied by the function S, where $S = H/H_1$, and H is the local stagnation enthalpy.† Thus S varies from 1 at the outer edge of the boundary layer to a value, at the wall, of h, the ratio of the wall temperature T_w to the equilibrium temperature T_e for zero heat transfer (for $Pr = 1$). Cooling the wall ($h<1$) would thus tend to lower the magnitude of the pressure gradient term, whereas heating ($h>1$) would tend to increase it. Thus conclusion 1 might be reached, and conclusion 2 would then be more or less expected as a corollary, since in an adverse pressure gradient cooling of the wall would tend to have a effect similar to that of diminishing this gradient. Another way[2,7] in which these conclusions have been reached is to note that when the momentum equation is integrated across the boundary layer (momentum integral equation) in conjunction with the Dorodnitsyn-Howarth transformation

$$y = \int_0^t \frac{T}{T_1}\, dt \tag{1}$$

the pressure gradient term appears in a form multiplied by a factor that decreases as the wall is cooled and increases as the wall is heated. This occurs with or without normal mass transfer (suction or injection) at the wall. Moreover, the momentum partial differential equation (with $\mu \sim T$) evaluated at the wall yields[23]

$$(u/u_1)_{\eta\eta} = -(T_\infty/T_1)(\rho_1/\rho_\infty)(T_w/T_1)(u_1'/u_\infty)\lambda - (T_\infty/T_1)\varphi\lambda^{1/2}(u/u_1)_\eta \tag{2}$$

Here, again, the pressure gradient term (u_1'/u_∞) appears in a form multiplied by the wall temperature. Thus, cooling of the wall may be expected to tend to diminish the direct effect of a pressure gradient (with or without mass transfer at the wall).

Physically, conclusions 1 and 2 are usually explained by the decreased density of the fluid due to heating (for example) of the wall, and by the consequent greater susceptibility of the fluid to a pressure gradient. In this connection, however, it should also be observed that heating of the wall not only decreases the inertia terms (proportional to ρ) in the momentum equation, but also the viscous term, more specifically the normal gradient of the shear stress. In particular, if $\mu \sim T$ and the transformation (1) is applied, it is found that the viscous term $\partial/\partial y(\mu \partial u/\partial y) = \mu_1(T_1/T)_{utt} = \mu_1 \times (\rho/\rho_1)u_{tt}$. Thus, the viscous term, being proportional to ρ, is similarly decreased by heating, so that *all* of the terms, other than the pressure gradient, in the momentum equation are decreased by heating, and this makes the pressure gradient *relatively* larger, or equivalently, the effect of the pressure gradient tends to be increased by heating. Moreover, it may be noted that at the wall itself the inertia terms vanish (for zero mass transfer) and that, hence, at the wall the pressure gradient is balanced only by the viscous term [cf. Eq. (2)].

Case of a Highly Cooled Wall ($T_w \to 0$)

In connection with conclusion 1 of the preceding section it is of interest to consider the effect of extreme cooling. For example, in the limit of $T_w \to 0$, would the effect of a pressure gradient be essentially eliminated? In particular, in an adverse pressure gradient, would separation then become impossible? It appears that *in general* the answer to these questions must be in the negative. For example, similarity solutions do exist[4] for which $T_w = 0$ while the shear at the wall is zero. (It should perhaps be noted, however, that in these solutions the pressure gradient at the leading edge is infinite). These solutions will be discussed in further detail in the next chapter. Moreover, although the use of fourth-degree profiles in connection with the Kármán-Pohlhausen method might lead one to infer that separation is impossible when $T_w = 0$ [cf., eg., Eq. (5) of Ref. 2], the use of an additional boundary condition at the wall leads to expressions for the coefficient proportional to $(\partial u/\partial y)_w$ which indicate that separation could still occur even if $T_w = 0$. [Cf., e.g., Eq. (59) in conjunction with Eq. (58) of Ref. 8 or Eqs. (14) or (21) of Ref. 7]. In this connection it may be noted that the equations developed in Ref. 7 or 8 would imply that separation is impossible when $T_w = 0$ *if* it were true that the heat transfer must be zero at the separation point.‡ However, there is no a priori reason why this should be so, since, for example, Reynolds' analogy is no longer valid, in general, in a pressure gradient. Moreover, a perusal of the similarity solutions (e.g., of Ref. 4) with heat transfer indicates that the Nusselt number Nu is generally not zero when the skin-friction coefficient $C_f = 0$. [This, as also noted in Refs. 12 and 22, indicates a weakness in methods (e.g., Refs. 12 and 24) that assume the temperature as a simple (everywhere differentiable) function of the velocity u. For in that case, $(\partial T/\partial y) = 0$ where $\partial u/\partial y = 0$, so that Nu would be zero where $C_f = 0$.]

One quick means of seeing that, even with $T_w = 0$ the direct effect of a pressure gradient is not eliminated, is to note that after the Illingworth-Stewartson transformation the pressure gradient in the transformed incompressible plane is multiplied by a factor S, which, although it does indeed vanish at the wall when $T_w = 0$, varies, in a direction normal to the wall, to a value of unity at the outer edge of the boundary layer. Thus the pressure gradient term in the momentum partial differential equation vanishes only at the wall, but (probably) not elsewhere. In this connection it is also of interest to note that, even if one restricts oneself to values at the wall, the pressure gradient term will not be entirely eliminated. Thus, considering for simplicity the case of an impermeable wall with $\mu \sim T$ and $Pr = 1$, the momentum partial differential equation evaluated at the wall yields [cf. Eq. (2)]§

$$(u/u_1)_{\eta\eta} = h(u_1'/u_\infty)\, F(\xi) \tag{3}$$

where $F(\xi) = -(T_\infty/T_1)(\rho_1/\rho_\infty)\lambda\{1 + [(\gamma - 1)/2]M_1^2\}$. As far as Eq. (3) is concerned, it would indeed seem that, if $h = 0$, then the pressure gradient term is eliminated. However, by differentiating the momentum partial differential equation with respect to t and evaluating terms at the wall, the following additional relation is obtained [Eq. (25) of Ref. 8]:

$$h(u/u_1)_{\eta\eta\eta} = (u/u_1)_{\eta\eta}(H/H_1)_\eta \tag{4}$$

If one now substitutes for $(u/u_1)_{\eta\eta}$ according to Eq. (3) into Eq. (4), one finds (upon cancelling the h's)

$$(u/u_1)_{\eta\eta\eta} = (u_1'/u_\infty)\, F(\xi)(H/H_1)_\eta \tag{5}$$

If the pressure gradient were zero, Eqs. (3) and (5) would imply

$$(u/u_1)_{\eta\eta} = 0 \tag{6}$$

$$(u/u_1)_{\eta\eta\eta} = 0 \tag{7}$$

whether h is zero or not. However, even with $h = 0$, but with a pressure gradient, although Eq. (6) would remain, Eq. (7) would no longer hold, but would have to be replaced

† In the literature, the symbol S often denotes $[(H/H_1) - 1]$.

‡ In that case, the coefficient b_1 appearing in Refs. 7 and 8 would vanish at the separation point.

§ It is to be understood in Eqs. (3–7) above [as well as in Eq. (2)] that all derivatives with respect to η are to be evaluated at the wall.

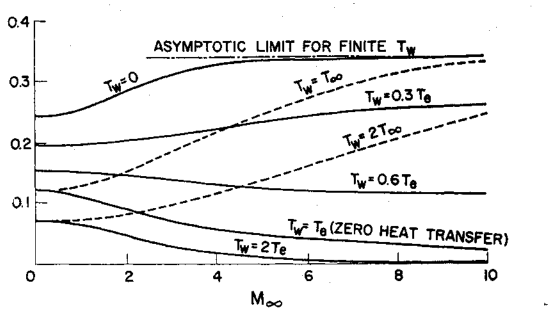

Fig. 1 The variation of $\xi_s \equiv (u_\infty - u_1)/u_\infty$ at separation with Mach Number and wall temperature for the linear adverse velocity gradient case (from Ref. 10).

by Eq. (5), where the pressure gradient term remains.¶ Thus, even with $h = 0$, the pressure gradient term is not entirely eliminated.

Finally, it is of interest to note Fig. 1 here,** taken from Gadd's paper.[10] The top curve gives the value of ξ at separation for the case $u_1/u_\infty = 1 - \xi$, as a function of M_∞, when $T_w = 0$. According to the curve, separation still occurs in this case of extreme cooling. An independent calculation of this case of $T_w = 0$ for various Mach numbers would be of interest.

Conditions at Separation

Since the series solution of Howarth,[28] the theoretical analysis of Goldstein,[22] and the machine solution of Leigh,[30] it is well known that a singularity at the separation point ($\tau_w = 0$) occurs in the exact solution of the boundary-layer equations for the case of incompressible flow and a linearly decreasing external velocity. More generally, it appears that for compressible flows with zero heat transfer a consistent expansion for the stream function ψ near the separation point can be made either on the assumption that it is analytic or that it is no longer analytic there.[26] In the latter case, it is found that, close to the separation point,

$$\tau_w \sim (X_s - X)^{1/2} \tag{8}$$

where X denotes distance in the transformed incompressible plane. A very quick means of surmising Eq. (8) would be to note that, if the momentum partial differential equation of the incompressible boundary layer over an impermeable wall is differentiated twice with respect to y, and the terms are evaluated at the wall, then it is found that at the wall

$$u_y(\partial u_y/\partial x) = (\mu/\rho)(\partial^4 u/\partial y^4) \tag{9}$$

Thus, *if* $(\partial^4 u/\partial y^4)_w \neq 0$ at the separation point, i.e., where $(u_y)_w = 0$, then integrating Eq. (9) with respect to x near the separation point ($x = x_s$) leads to Eq. (8). Equation (8) in conjunction with the continuity equation will be found to imply that the normal velocity component v at the separation point will become infinite (except at $y = 0$):

$$v \sim (x_s - x)^{-1/2} \tag{10}$$

Thus, it appears that, if one *requires* $v \to \infty$ at the separation point, then $(\partial^4 u/\partial y^4)_w \neq 0$ there, and Eq. (8) must hold. Landau,[45] in fact, has started from the requirement of $v \to \infty$ at separation, and has inferred therefrom Prandtl's condition $(u_y)_w = 0$ at separation, together with equations of the form (8) and (10).

In the case of heat transfer at the wall, however, Stewartson[26,31] has shown that the boundary layer is regular at the point ($X = X_s$) where $\tau_w = 0$ and has inferred that the center of the singularity in this case is in the interior of the boundary layer. In particular, the singularity lies upstream of $X = X_s$ if the wall is cooled and presumably downstream of $X = X_s$ if the wall is heated. Stewartson therefore inferred that actual separation may no longer occur exactly where $\tau_w = 0$ and that if the wall is cooled it may occur earlier.††

In connection with the singularity at the separation point, it is perhaps worthwhile to observe that, in treating the case of a linearly diminishing external velocity in incompressible flow by means of a momentum integral method, the satisfaction by the velocity profiles of a condition at the wall obtained by differentiating the momentum partial differential equation twice with respect to normal distance y and taking values at the point where $\tau_w = 0$ has led to greatly improved accuracy in predicting the separation point.[32,33] However, a term $[(\partial u/\partial y)\cdot\partial/\partial x(\partial u/\partial y)]_w \sim \tau_w(\partial\tau/\partial x)_w$, which appears here, was set equal to zero. According to Eq. (8), such a term will actually not vanish at the separation point ($X = X_s$) in an exact solution. In spite of this, good agreement with exact solutions for a variety of flows has thereby been obtained for the location of the separation point.[34] One possible explanation of this would appear to be that, in the exact solutions, the singularity makes itself felt only extremely close to the separation point, so that the quantity $\tau_w\ (\partial\tau/\partial x)_w$ is actually close to zero fairly near, although not exactly zero at, the separation point.

The entire preceding discussion has been based on the classical boundary-layer assumptions and equations. Although the present review is concerned only with this classical framework, it may be worthwhile here to discuss briefly the conditions at the separation point according to physically more exact analyses.‡‡ First, it is generally agreed that actual flows will not exhibit any singularities at separation and that, hence, the preceding implications of the *classical* boundary-layer equations indicate that the latter are no longer valid at separation. According to the Navier-Stokes equations, it is possible to obtain the nature of solutions without singularities near the separation point at the wall.[46,47] The analysis of Oswatitsch[47] is remarkably simple, and appears worthwhile to review here. *Assuming* that the velocity components u and v can be expressed in a Taylor series about the separation point ($x = y = 0$) at the wall and noting that all x derivatives of u vanish at the wall, whereas for an impermeable wall the continuity equation implies $v = v_y = v_{xy} = 0$ at the wall,

$$u = u_y y + u_{xy}\, xy + 1/2\, u_{yy} y^2 + \ldots \tag{11}$$

$$v = 1/2\, v_{yy}\, y^2 + \ldots \tag{12}$$

where all derivatives are evaluated at $x = y = 0$. Moreover, at the wall ($y = 0$) the continuity equation implies

$$v_{yy} = -u_{xy} - (\rho_x/\rho)u_y \tag{13}$$

Hence, the streamlines will be given by

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{1}{2}\frac{[u_{xy} + (\rho_x/\rho)\, u_y]\, y}{u_y + u_{xy}\, x + \frac{1}{2}\, u_{yy} y} \tag{14}$$

From Eq. (14) it is seen that, in order that a separation streamline (i.e., a streamline other than $y \equiv 0$) occur at $x = y = 0$, it is necessary that $u_y = 0$. Thus Prandtl's condition

¶ This, in fact, explains why, as previously noted, the use of fourth-degree profiles satisfying condition 3 without condition 4 implies that separation would be impossible with $T_w = 0$, whereas the use of higher degree profiles satisfying both conditions 3 *and* 4 no longer implies such an impossibility.

** In Gadd's words, this figure is meant to show curves, which an "exact integration of the boundary-layer equations . . . would probably give." They show "the general shape of the curves with a very rough indication of the numerical values."

†† In the succeeding (as well as in the preceding) section here, however, the "separation point" is to be understood to mean the point at the wall where $\tau_w = 0$.

‡ The author is indebted to Lester Lees for his valuable discussions on this matter in a private communication.

is derived. Since the shear stress at the wall will be $\tau = \mu u_y$, whereas the Navier-Stokes equation in the x direction will imply $p_x = \mu u_{yy}$ at the separation point at the wall, the streamlines near separation (both in the upstream and downstream regions) will then be given by

$$dy/dx = -[\tau_x y/(2\tau_x x + p_x y)] \quad (15)$$

Setting $y = x \tan\theta$, where θ is the angle that the separation streamline makes with the wall, Eq. (15) yields

$$\tan\theta = -3(\partial\tau/\partial x)/p_x \quad (16)$$

From the Navier-Stokes equation in the y direction, it will be found that at the separation point the normal pressure gradient will be

$$p_y = \mu\, v_{yy} = -\partial\tau/\partial x = \tfrac{1}{3}(\tan\theta)p_x \quad (17)$$

Finally, Eq. (15) implies that the locus, near the separation point, of points where the streamlines will have vertical slopes (in the reverse-flow region), i.e., where $u = 0$, will be

$$y = -2(\tau_x/p_x)x = \tfrac{2}{3}(\tan\theta)x \quad (18)$$

It is important to observe that this analysis holds quite generally for compressible steady two-dimensional flows over an impermeable wall with or without heat transfer.§§

The preceding analysis, however, is still not quite complete, since $(\partial\tau/\partial x)$ and p_x remain undetermined there. In classical boundary-layer theory, p_x is determined by being prescribed, whereas $\partial\tau/\partial x$ is determined by solutions upstream of the separation point. For the case $u_1/u_\infty = 1 - \xi$, this yields, according to Eq. (8), $\partial\tau/\partial x \to -\infty$ [thus really contradicting the assumptions governing Eqs. (11) and (12)]. Another important class of problems in which such quantities are determined is that in which the boundary layer interacts with the external flow, for example laminar separation in a supersonic stream induced by a disturbance downstream, such as a shock wave. References 48 and 49 are quite recent examples of theoretical analyses of such problems. The boundary-layer approximations are still used there, but the external flow is no longer considered as rigidly prescribed, but as determined through its interaction with the boundary layer near separation. For classical boundary-layer theory in which p_x (positive) is fixed, it may be noted that if $\partial\tau/\partial x \to -\infty$ at separation, then Eq. (16) implies that the separation streamline will be vertical. This verifies *formally* the boundary-layer result for the case $u_1/u_\infty = 1 - \xi$, in which $v \to \infty$ at separation. In this case, as can be expected from Eq. (8), it is impossible to obtain solutions of the boundary-layer equations *across* the separation point.[29] However, regular solutions across and beyond the separation point are obtained in the interaction analyses of Refs. 48 and 49. One simple *necessary* test of whether the boundary-layer approximations in the interaction problems might remain approximately valid is to check whether Eqs. (16–18) are approximately verified *and* whether $\tan\theta$ is now small so that v is, in some adequate sense, "small," and p_y is negligible. Reference 49, for example, has to some extent considered this explicitly. Equation (18) was approximately verified, and $|3(\partial\tau/\partial x)/p_x|$ was of order 0.13. Further investigation is probably necessary to establish definitely whether, or to what degree, the boundary-layer approximations are valid in these interaction problems.

It is, finally, of interest to note that, for incompressible flow, Meksyn[50–52] found that, according to his asymptotic approximate method of solving the boundary-layer equations, a regular solution near the separation point (including the downstream region) could be obtained only if $d^2u_1/dx^2 > 0$ near separation. Meksyn emphasized that the "external" flow near separation must in actual cases be modified by the boundary layer in such a way that the actual flow will pass smoothly through the separation point.¶¶

§§ Oswatitsch[47] also analyzes separation conditions in three-dimensional flows.

IV. Specific Numerical Solutions

The various numerical solutions that have been obtained relevant to the effect of heat transfer on laminar separation will now be reviewed. Among other solutions, these include similarity solutions and those for a linearly retarded external velocity. The case of a permeable wall and the effect of Prandtl Number and viscosity-temperature relation will also be briefly discussed.

Similarity Solutions

By far, most of the solutions on the compressible laminar boundary layer with heat transfer which may be considered as exact are the similarity solutions[4–6] in which $U_1 \sim X^m$ and adverse pressure gradients are represented by negative values of m.

In connection with the effect of heat transfer on separation, the results of the similar solutions of primary interest here are probably the values of m, as a function of wall temperature, required for a zero-skin-friction boundary layer. These are shown in Table 1. It is noted that, as the wall is cooled, a larger negative m, corresponding to a larger adverse pressure gradient, is required for separation.* This may be considered to illustrate the tendency of cooling to delay or prevent separation. The entry for $T_w/T_\infty = 0$ illustrates the possibility of separation in such a case.

Solutions for the Case $u_1/u_\infty = 1 - \xi$

This case may almost be regarded as the prototype of an adverse pressure gradient. It has probably been the most frequently studied case of an adverse pressure gradient. The various results for zero heat transfer in this case have been summarized in the Appendix. For the cases of heat transfer, interest may be focused here on the separation point ξ_s as a function of wall-temperature ratio and Mach number M_∞.

Table 1 Similarity solutions ($U_1 \sim X^m$)

T_w/T_∞	Values of m for $\tau_w = 0$	Refs.
2	−0.06	4, 6
1	−0.094	6
0.6	−0.109	4
0.5	−0.1178	6
0.25	−0.1351	6
0.2	−0.134	4
0	−0.1400	4

¶¶ Meksyn's actual condition for regular flow at separation, based on his approximate asymptotic method and on his study of the flow over an elliptic cylinder using the external velocity distribution as actually measured by Schubauer, was that the parameter $\lambda(\alpha)$ have a maximum, with separation occurring downstream of that maximum:

$$\left[\lambda \equiv -2\left(\frac{\alpha}{u_1}\right)\frac{du_1}{d\alpha} \qquad \alpha \equiv \int_0^x u_1\, dx\right]$$

This condition will imply $u_1''(x) > 0$ near separation. It might also be mentioned that, by his method, Meksyn actually had difficulty reaching the separation point from the upstream region but found it easier to approach it from the downstream region. Of course, the boundary-layer equations alone would be insufficient to determine $u_1(x)$ near or beyond separation, since additional matching conditions between the external flow and the boundary layer would be needed (cf., e.g., Refs. 48 and 49).

* A comparison between the entries for $T_w/T_\infty = 0.20$ and 0.25 slightly contradicts this trend, but this may be due to the use of $Pr = 1$ in Ref. 4 and $Pr = 0.7$ in Ref. 6.

A number of investigators, by a variety of methods, have considered this problem, and the results are summarized in Table 2.

In addition to Table 2, attention is also called to the sketched curves of ξ_s vs M_∞ for the extreme cooling case $T_w = 0$ in Ref. 10 (Fig. 1 herein) and to the fact that, in general, the curves of ξ_s vs M_∞ for a fixed wall-to- freestream temperature ratio T_w/T_∞ will have different shapes from those for a fixed wall-to-equilibrium temperature T_w/T_e (Fig. 1 herein, also Ref. 7, p. 17).

By reading a given column, for a given M_∞, downward in Table 2, it is seen that all theories agree that, for a given Mach number, lowering the wall temperature moves the separation point downstream (cf. also Fig. 1 herein). As previously indicated, an independent calculation for the case of $T_w = 0$, or low T_w/T_e, would be definitely worthwhile. It may be remarked that the curve for $T_w/T_\infty = 2$ of Gadd (Fig. 1 herein) is quite similar, even quantitatively, to that of Morduchow and Grape (Fig. 6 in Ref. 7). There appears to be on the whole good agreement among all investigators for the low-speed case $M_\infty = 0$. Finally, it may be remarked that Illingworth[3] inferred that ξ_s should depend primarily on T_w/T_∞ and be almost independent of M_∞, but this is apparently contradicted by the curves in both Refs. 7 and 10 [cf. Fig. 1 herein].

Solutions for Other Cases

In addition to the similarity solutions and the various solutions for the case $u_1/u_\infty = 1 - \xi$, there have been a few other cases for which the effect of heat transfer on laminar separation has been calculated. Poots[13] considered the case $U_1/U_\infty = 1 - \frac{1}{8}X$, where U and X are, respectively, the longitudinal velocity component and the distance variable in the incompressible plane according to the Stewartson transformation. He calculated the case of zero heat transfer ($T_w/T_e = 1$) and of a heated wall ($T_w/T_e = 2$). The results for various Mach numbers showed an upstream movement of the separation point when the wall was heated. Morduchow and Grape[7] have considered the case in which a stagnation flow is followed by an adverse pressure gradient and have calculated the adverse pressure gradient required for "immediate" separation, as a function of the wall temperature. By means of the analysis of Morduchow and Grape, Gadd[10] gives a corresponding result if the initial region is one of zero pressure gradient, instead of a stagnation flow. Baxter and Flügge-Lotz[11] have calculated somewhat similar cases, in which a zero pressure gradient is followed by either a step pressure gradient or a ramp pressure gradient, and found in each case that separation would occur sooner with a hotter wall. Curle[12] and Luxton and Young[15] have considered the case of a linear pressure rise, i.e., a constant adverse pressure gradient, namely $p = p_\infty(1 + \xi)$. For the case $M_\infty = 2$ and $T_w/T_\infty = 1$ (cooled wall), Curle obtained separation at $\xi = 1.32$, Luxton and Young found $\xi_s = 1.04$, and the Mathematics Division of the National Physical Laboratory obtained $\xi_s = 1.25 \pm 10\%$. Walz[24] has calculated the case $u_1/u_\infty = \xi(5 + \xi)/(1 + \xi)^2$ and found that separation is delayed by cooling. (Actually C_f was found to come very close to zero at a fairly early station when the wall was cooled, but then C_f increased.) Walz also considered the case of $u_1/u_\infty = 1 + \epsilon \sin(2\pi p \xi)$, which had been previously considered by Dryden and others in connection with the onset of turbulence. Walz shows that the minimum value of ϵ required to lead to separation is appreciably increased by wall cooling. Fannelöp and Flügge-Lotz[43] have quite recently calculated the boundary layer over a flat-plate leading-edge section followed by a semi-infinite wavy wall and found that separation occurred earlier for a heated wall than for an adiabatic wall, whereas cooling considerably delayed separation.

Permeable Wall

In spite of the substantial literature on the boundary layer over a porous wall (suction or injection), the only work on porous walls with heat transfer concerned specifically with laminar separation seems to have been by Morduchow and co-workers.[2,9,23] Normal fluid injection tends to promote separation (as shown, e.g., in Ref. 2), whereas cooling of the wall, per se, tends to delay separation. The simultaneous effect of injection and wall cooling is considered in Ref. 2, which is based on fourth-degree profiles, and in Ref. 9, which is based on seventh-degree velocity profiles. In Ref. 9, a heat-balance equation was taken into account, so that the injection mass-flow was related to the wall temperature and the coolant temperature T_c. As an example, a plot of the separation point as a function of T_w/T_e and T_c/T_e for the case $u_1/u_\infty = 1 - \xi$ and various Mach numbers showed that the net effect of the simultaneous injection and wall cooling

Table 2 Calculated separation points for the case $u_1/u_\infty = 1 - \xi$

		Value of ξ_s according to several authors										
M_∞	$\frac{T_w}{T_\infty}$	Gadd[1]	Illingworth[3]	Morduchow and Grape[7]	Gadd[10]	Curle[12]	Luxton and Young[15]	Poots[13]	Monaghan[17]	Savage[22]	Math. Div. Natl. Phys. Lab.	Libby and Fox[44]
0	2		0.067	0.073	0.072	0.071		0.075		0.075		
	1.295		0.093	0.106								
	0.8		0.128	0.135								
	0.6			0.152	0.16							
	0.5			0.168		0.195						
	0.3			0.190	0.195							
	0.238									0.223		
1	2			0.077	0.077							
	1				0.128							
	0.96		0.118									
$(10)^{1/2}$	2.4		0.0729									
	2			0.105	0.10							
	1.5			0.124^a								
	1		0.117		0.185							
4	2			0.123	0.118							
	1	0.25		0.173^b	0.21	0.311	0.177		0.175^c		0.227	0.1065
									0.22^d		(±10%)	

^{a} From Fig. 9 of Ref. 23.
^{b} Calculated separately.
^{c} Based on taking $m_{sep} = 0.034$.
^{d} Based on taking $m_{sep} = 0.042$.

Table 3 Calculated separation points $\xi = \xi_s$ for $u_1/u_\infty = 1 - \xi$ with zero heat transfer

	M_∞						
	0	1	3	3.16	4	6	10
Stewartson[19]	0.120	0.110	0.077		0.062	0.044	0.024
Morduchow and Clarke[33]	0.122	0.113	0.077		0.062	0.042	0.023
Thwaites-Stewartson[35]	0.120	0.111			0.060	0.043	0.023
Wrage[37]	0.125	0.112	0.072		0.056	0.037	0.021
Curle[12]					0.067		
Tani[14] $T_\infty = 72°R$	0.120	0.111	0.081		0.066	0.048	
$T_\infty = 648°R$	0.120	0.112	0.085		0.072	0.057	
Luxton and Young[15]					0.064		
Gadd[10] (see Fig. 1 herein)	0.120	0.106	0.073		0.062	0.046	0.023
Math. Div., Natl. Phys. Lab.[12]					0.045 (±10%)		
Gadd[1]	0.114				0.049		
Illingworth[3]	0.111	0.101		0.0602			0.0142
Loftin and Wilson[38]	0.102				0.028	0.014	0.007
Howarth[39]	0.156	0.148		0.107			0.052
Young[36]	0.165	0.148		0.081			0.013
Libby and Fox[44]	0.120		0.055				0.007

was to move the separation point upstream. In Ref. 23 on the boundary layer with suction, calculations were included to indicate the effect of wall cooling on the separation point for the case $u_1/u_\infty = 1 - \xi$ with suction. The same type of approximation for the heat-transfer coefficient $b_1(\xi)$ in the stagnation enthalpy was made here (and in Ref. 9) as in Ref. 7. In connection with heat transfer, another problem of interest here would be to determine, as a function of wall temperature, the minimum ("critical") value of the suction parameter φ required to entirely avoid separation in the case $u_1/u_\infty = 1 - \xi$. This has actually been carried out in Ref. 23 for $M_\infty = 0$, but because of the important role played by the b_1 terms in the case of heat transfer in this particular problem, the results must at present be regarded as quite tentative. One very simple conclusion that can be stated in connection with the effect of wall cooling on separation with suction is that, since cooling of the wall leads to an increased density ρ_w of the fluid there, the suction parameter φ will thereby be increased; hence, the effect of a given suction *velocity* v_w will be increased, and separation will be delayed by virtue of this effect alone. In fact, with sufficient cooling, the suction parameter could thereby be made sufficiently large to entirely prevent separation (cf. Ref. 23).

Effect of Prandtl Number and Viscosity-Temperature Relation

Virtually all of the specific results included here have been based on a Prandtl number Pr of 1 and a viscosity coefficient proportional to the absolute temperature. For air, of course, these assumptions are not quite valid (e.g., $Pr \approx 0.72$), although they have usually been found to give very useful approximations. Prandtl number and viscosity-relation effects have been considered in a variety of investigations, but the concern here is specifically with such effects on laminar separation. In this connection, Refs. 1, 15, 16, and 35 may be cited. The effects of Prandtl number and of viscosity-temperature relation on laminar separation have usually been regarded as comparatively small. For example, in his analysis for the case of zero heat transfer Gadd[35] concludes that, for Mach numbers up to 10, the use of $Pr = 1$ and $\mu \sim T$ for air introduces "no serious errors . . . into the calculation of the laminar separation point." It may be noted that, if one uses the Chapman-Rubesin type of modification of $\mu \sim T$, namely, $\mu = CT$, where C is chosen to satisfy, e.g., the Sutherland relation at the wall, then it is found[7,16] that, when the wall temperature is uniform so that C will be a constant, the value of C will not affect the position of laminar separation. In general, in spite of some quantitative differences, all analyses agree that the assumption $Pr = 1$ instead of $Pr = 0.7$ leads to an earlier separation point, whereas the assumption of $\mu \sim T$ instead of $\mu \sim T^\omega$, where $\omega<1$, also leads to an earlier separation. Moreover, these effects increase with Mach number. For example, for the case $u_1/u_\infty = 1 - \xi$ and zero heat transfer, Gadd[35] finds that, for $\omega = 1$ and Pr near 1,

$$(\xi_s - \xi_{s1})/\xi_{s1} = k(1 - Pr) \tag{19}$$

where ξ_{s1} is the value of ξ_s for $Pr = 1$, and $k \approx 0.27$ when $M_\infty \approx 4$, whereas $k = 0.9$ when $M_\infty = 10$. Gadd estimates that by Young's method[36] the values of k would be 0.28 and 0.41, respectively. Moreover, with $Pr = 1$ but ω different from 1, Gadd finds

$$(\xi_s - \xi_{s1})/\xi_{s1} = k'(1 - \omega) \tag{20}$$

where $k' = 0.18$ when $M_\infty = 4$, and $k' = 0.53$ when $M_\infty = 10$. For the case $u_1/u_\infty = 1 - \xi$ and $M_\infty = 4$ with zero heat transfer, Luxton and Young have, more recently, found $\xi_s = 0.078$ when $\omega = 0.89$ and $Pr = 0.725$, in comparison with $\xi_s = 0.064$ when $\omega = Pr = 1$ (cf. Fig. 12 of Ref. 15). For heat transfer, Curle[16] has concluded that the effect of Prandtl number increases as the wall is cooled. For $u_1/u_\infty = 1 - \xi$, $M_\infty = 4$, and $T_w/T_\infty = 1$ (cooled wall), Curle finds that a change of Pr from 1 to 0.73 moves the separation point downstream by about 7%.

Appendix

In view of the many independent calculations that have been carried out to determine the effect of Mach number on the laminar separation point in the case $u_1/u_\infty = 1 - \xi$ with zero heat transfer, it appears worthwhile to briefly summarize these results. The results of the various investigations for the separation point $\xi = \xi_s$ are shown in Table 3.

In Table 3, all entries are for $Pr = 1$ and $\mu \sim T$, with the exception of Tani, who used $Pr = 0.72$ and the Sutherland viscosity-temperature relation. In connection with Tani's results, reference may be made, for example, to Eqs. (19) and (20), according to which Tani's results corrected for $Pr = 1$ and $\mu \sim T$ would have to be moved somewhat upstream.†

In connection with Table 3 it should perhaps first be noted that the value of ξ_s, namely, $\xi_s = 0.120$, for $M_\infty = 0$ is now generally accepted as exact. At nonzero Mach numbers, however, there do not at present appear to be any values that have been universally accepted as exact. The value of ξ_s at $M_\infty = 4$ obtained by the Mathematics Division of the National Physical Laboratory (NPL) (namely, 0.045 ±

† For the case of $Pr = 1$ and $\mu \sim T$, calculations by Tani's method have just been completed by James Bentson with the following results: $\xi_s = 0.109, 0.061, 0.043$, and 0.024 for $M^\infty = 1$, 4,6, and 10, respectively.

10%) has been based on a numerical solution of the original partial differential equations and was reported by Curle[12] in 1958. However, in view of the fact that no further details on the method of solution were given, and that the result disagrees with most values that have been obtained by other methods (cf. especially the first eight entries in the table which seem to be in essential agreement with one another), the NPL value must at present be regarded as in doubt.

Despite the fairly large scatter, for any given Mach number, in the value of ξ_s shown in the table, there are a number of items of agreement there. In particular, all theories agree that an increase in Mach number moves the separation point upstream. Moreover, the results in the first eight references appear to be in essential *quantitative* agreement at all Mach numbers.

A brief summary of the methods used by the various investigators may be worthwhile. Stewartson's values are based on an application of his transformation in conjunction with the approximate polygonal method of Howarth.[28] The Morduchow-Clarke values were obtained by a momentum integral method in conjunction with seventh-degree velocity profiles satisfying additional conditions at the wall not satisfied by the fourth-degree profiles in the usual Kármán-Pohlhausen method. The values obtained by Gadd[35] in 1953 and by Curle[12] are essentially the result of an application of Stewartson's transformation to the method of Thwaites[21] originally developed for incompressible flow. As previously indicated, Tani's values are based on the use of a two-parameter integral method. Wrage obtained his values by extending the series method of Görtler[40] to compressible flows. Luxton and Young have modified Young's early method[36] for greater accuracy, to yield their "complete" method, with the results shown in row 7. The earliest values appear to have been Howarth's,[39] based on the Kármán-Pohlhausen method with the usual type of fourth-degree profiles. Loftin and Wilson's results are based upon an application of Stewartson's transformation to a comparatively early method of Von Doenhoff.[41] Illingworth's values are based on a series method, which appeared to become particularly less reliable at high Mach numbers because of inadequate convergence. Finally, Libby and Fox's values are based on a two-parameter momentum, and moment of momentum, integral in conjunction with Lees-Levy variables and the use of a high-speed computer.

In addition to the entries listed in Table 3, it appeared worthwhile to determine the implications of Stewartson's transformation applied to the Stratford-Curle-Skan criterion,[42] which for incompressible flow has yielded very good results for a variety of cases. This criterion, for incompressible flow, states that separation occurs at that value of ξ for which

$$C_p[\xi(dC_p/d\xi)]^2 = 0.0104 \tag{A1}$$

where C_p is the pressure coefficient, given by

$$C_p = 1 - (u_1/u_\infty)^2 \tag{A2}$$

For compressible flow with zero heat transfer, $Pr = 1$ and $\mu \sim T$, the Illingworth-Stewartson transformation applied to Eqs. (A1) and (A2) yields

$$\bar{C}_p[\bar{\xi}(d\bar{C}_p/d\bar{\xi})]^2 = 0.0104 \tag{A3}$$

where

$$\bar{C}_p = 1 - (U_1/u_\infty)^2 \qquad U_1 = (T_1/T_\infty)^{-1/2}u_1 \tag{A4}$$

$$\bar{\xi} = \int_0^\xi \left(\frac{T_1}{T_\infty}\right)^{(3\gamma-1)/2(\gamma-1)} d\xi \tag{A5}$$

Moreover,

$$T_1/T_\infty = 1 + [(\gamma-1)/2]M_\infty^2[1 - (u_1^2/u_\infty^2)] \tag{A6}$$

Equation (A3) in conjunction with Eqs. (A4–A6) is found to become

$$\left(1 - \frac{u_1^2}{u_\infty^2}\right)\left(\frac{u_1}{u_\infty}\right)^2\left(\frac{u_1'}{u_\infty}\right)^2\left[\int_0^\xi \left(\frac{T_1}{T_\infty}\right)^{(3\gamma-1)/2(\gamma-1)} d\xi\right]^2 \times \left(\frac{T_1}{T_\infty}\right)^{-[2(4\gamma-3)/(\gamma-1)]} = \frac{0.00260}{\{1 + [(\gamma-1)/2]M_\infty^2\}^3} \tag{A7}$$

For $u_1/u_\infty = 1 - \xi$ and $\gamma = 1.4$, solutions of Eq. (A7) for ξ for various Mach numbers yield $\xi_s = 0.121, 0.114$, and 0.089 for $M_\infty = 0,1$, and 3, respectively. For $M_\infty \geq 4$, no (physical) roots of Eq. (A7) are obtainable.‡ Thus it appears that the criterion (A1) can be extended reliably to compressible flows with zero heat transfer by the Illingworth-Stewartson transformation only for low Mach numbers ($M_\infty \leq 1$).

References

[1] Gadd, G. E., "The numerical integration of the laminar boundary layer equations, with special reference to the position of separation when the wall is cooled," Aeronautical Research Council Paper 312 (August 1, 1952).

[2] Morduchow, M. and Galowin, L., "The compressible laminar boundary layer in a pressure gradient over a surface cooled by fluid injection," *Proceedings of the Iowa Thermodynamics Symposium* (State Univ. of Iowa, Iowa City, Iowa, 1953), pp. 143–169.

[3] Illingworth, C. R., "The effect of heat transfer on the separation of a compressible laminar boundary layer," Quart. J. Mech. Appl. Math. VII, 8 (1954).

[4] Cohen, C. B. and Reshotko, E., "Similar solutions for the compressible laminar boundary layer with heat transfer and pressure gradient," NACA TN 3325 (February 1955).

[5] Li, T. Y. and Nagamatsu, H. T., "Similar solutions of compressible boundary layer equations," J. Aeronaut. Sci. **22**, 607 (1955).

[6] Livingood, J. N. B. and Donoughe, P. L., "Summary of laminar-boundary layer solution for wedge-type flow over convection and transpiration cooled surfaces," NACA TN 3588 (December 1955).

[7] Morduchow, M. and Grape, R. G., "Separation, stability and other properties of compressible laminar boundary layer with pressure gradient and heat transfer," NACA TN 3296 (May 1955).

[8] Libby, P. A. and Morduchow, M., "Method for calculation of compressible laminar boundary layer with axial pressure gradient and heat transfer," NACA TN 3157 (January 1954).

[9] Morduchow, M., "Laminar separation over a transpiration-cooled surface in compressible flow," NACA TN 3559 (December 1955).

[10] Gadd, G. E., "A review of theoretical work relevant to the problem of heat transfer effects on laminar separation," Aeronautical Research Council Paper 331 (June 13, 1956).

[11] Baxter, D. C. and Flügge-Lotz, I., "The solution of compressible laminar boundary layer problems by a finite difference method," Stanford Univ., Div. of Engineering Mechanics, TR 110 (October 15, 1957).

[12] Curle, N., "The steady compressible laminar boundary layer, with arbitrary pressure gradient and uniform wall temperature," Proc. Roy. Soc. (London) **A249**, 206–224 (1958).

[13] Poots, G., "A solution of the compressible laminar boundary layer equations with heat transfer and adverse pressure gradient," Quart. J. Mech. Appl. Math. XIII, 57 (1960).

[14] Tani, I., "On the approximate solution of the laminar boundary-layer equations," J. Aeronaut. Sci. **21**, 487–495, 504 (1954).

[15] Luxton, R. E. and Young, A. D., "Generalized methods for the calculation of the laminar compressible boundary-layer characteristics with heat transfer and non-uniform pressure distribution," British Aeronautical Research Council R & M 3233 (January 1960).

[16] Curle, N., "The effects of heat transfer on laminar boundary-layer separation in supersonic flow," Aeronaut. Quart. XII, 309 (November 1961).

‡ The author acknowledges the aid of James Bentson in these calculations.

[17] Monaghan, R. J., "Effects of heat transfer on laminar boundary-layer development under pressure gradients in compressible flow," British Aeronautical Research Council R & M 3218 (May 1960).

[18] Illingworth, C. R., "Steady flow in the laminar boundary-layer of a gas," Proc. Roy. Soc. (London) **A199,** 533–558 (December 1949).

[19] Stewartson, K., "Correlated incompressible and compressible boundary layers," Proc. Roy. Soc. (London) **A200,** 84–100 (December 1949).

[20] Cohen, C. B. and Reshotko, E., "The compressible laminar boundary layer with heat transfer and arbitrary pressure gradient," NACA TN 3326 (April 1955).

[21] Thwaites, B., "Approximate calculation of the laminar boundary layer," Aeronaut. Quart. **1,** 245–280 (November 1949).

[22] Savage, S. B., "The effect of heat transfer on separation of laminar compressible boundary layers," Firestone Flight Sciences Lab., California Institute of Technology, Pasadena, Calif., Separated Flows Project, TR 2 (June 1962).

[23] Morduchow, M. and Reyle, S. P., "On the compressible laminar boundary layer with suction," Polytechnic Institute of Brooklyn Aerodynamics Lab., PIBAL Rept. 807 (October 1963).

[24] Walz, A., "Beitrag zur Näherungstheorie kompressibler laminarer Grenzschichten mit Wärmeübergang," Deutsche Versuchsanstalt für Luft-und Raumfahrt E.V., Bericht 281 (October 1963).

[25] Li, T. Y. and Nagamatsu, H. T., "The effect of heat transfer on the behavior of a compressible laminar boundary layer under favorable and adverse pressure gradients," J. Aeronaut. Sci. **21,** 850 (1954).

[26] Stewartson, K., *The Theory of Laminar Boundary Layers in Compressible Fluids* (Oxford University Press, New York, 1964), Chap. 4.

[27] Low, G. M., "The compressible laminar boundary layer with heat transfer and small pressure gradient," NACA TN 3028 (October 1953).

[28] Howarth, L., "On the solution of the laminar boundary layer equations," Proc. Roy. Soc. (London) **A164,** 547–579 (February 1938).

[29] Goldstein, S., "On laminar boundary-layer flow near a position of separation," Quart. J. Mech. Appl. Math. **1,** 43–69 (1948).

[30] Leigh, D. C. F., "The laminar boundary-layer equation: A method of solution by means of an automatic computer," Proc. Cambridge Phil. Soc. **51,** 320–332 (1955).

[31] Stewartson, K., "The behavior of a laminar compressible boundary layer near a point of zero skin-friction," J. Fluid Mech. **12,** 117–128 (1962).

[32] Timman, R., "A one parameter method for the calculation of laminar boundary layers," Nationaal Luchtvaartlaboratorium Rept. F35 (1949).

[33] Morduchow, M. and Clarke, J. H., "Method for calculation of compressible laminar boundary-layer characteristics in axial pressure gradient with zero heat transfer," NACA TN 2784 (September 1952).

[34] Morduchow, M. and Reyle, S. P., "On calculation of the laminar separation point, and results for certain flows," J. Aerospace Sci. **29,** 996–997 (1962).

[35] Gadd, G. E., "Some aspects of laminar boundary layer separation in compressible flow with no heat transfer to the wall," Aeronaut. Quart. **IV,** 123–150 (February 1953).

[36] Young, A. D., "Skin friction in the laminar boundary layer in compressible flow," Aeronaut. Quart. **1,** 137–164 (August 1949).

[37] Wrage, E., "Entwicklung und Anwendung einer Allgemeiner Reihenmethode zur Berechnung laminarer, kompressibler Grenzschichten," Deutsche Versuchsanstalt für Luftfahrt E.V., Bericht 134 (1960).

[38] Loftin, L. K., Jr. and Wilson, H. B., Jr., "A rapid method for estimating the separation point of a compressible laminar boundary layer," NACA TN 2892 (February 1953).

[39] Howarth, L., "Concerning the effect of compressibility on laminar boundary layers and their separation," Proc. Roy. Soc. (London) **A194,** 16–42 (1948).

[40] Görtler, H., "A new series for the calculation of steady laminar boundary-layer flows," J. Math. Mech. **6,** 1–66 (1957).

[41] Von Doenhoff, A. E., "A method of rapidly estimating the position of the laminar separation point," NACA TN 671 (1938).

[42] Curle, N. and Skan, S. W., "Approximate methods for predicting separation properties of laminar boundary layers," Aeronaut. Quart. **VIII,** 257–268 (August 1957).

[43] Fannelöp, T. and Flügge-Lotz, I., "The laminar compressible boundary layer along a wave-shaped wall," Ing.-Arch., **XXXIII,** 24–35 (1963).

[44] Libby, P. A. and Fox, H., "A moment method for compressible laminar boundary layers and some applications," Polytechnic Institute of Brooklyn Aerodynamics Lab. PIBAL Rept. 615 (September 1964).

[45] Landau, L. D. and Lifshitz, E. M., *Fluid Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959), pp. 151–156.

[46] Dean, W. R., "Note on the motion of liquid near a position of separation," Proc. Cambridge Phil. Soc. **46,** 293–306 (1950).

[47] Oswatitsch, K., "Die Ablosungsbedingung von Grenzschichten," *Boundary Layer Research I.U.T.A.M. Symposium Freiburg,* edited by H. Görtler (Springer-Verlag, Berlin, 1957), pp. 357–367.

[48] Lees, L. and Reeves, B. L., "Supersonic separated and reattaching laminar flows: I. General theory and application to adiabatic boundary-layer/shock-wave interactions," AIAA J. **2,** 1907–1920 (1964).

[49] Nielsen, J. N., Lynes, L. L., Goodwin, F. K., and Holt, M., "Calculation of laminar separation with free interaction by the method of integral relations," AIAA Preprint 65-50 (1965).

[50] Meksyn, D., "Integration of the laminar boundary layer equation. I. Motion of an elliptic cylinder. Separation," Proc. Roy. Soc. (London) **A201,** 268–283 (1950).

[51] Meksyn, D., "Integration of the boundary-layer equations," Proc. Roy. Soc. (London) **A237,** 543–559 (1956).

[52] Meksyn, D., *New Methods in Laminar Boundary-Layer Theory* (Pergamon Press, New York, 1961), pp. 101–115.